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On Hermite Interpolation[1]

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More general and stronger estimations of bounds for the fundamental functions of Hermite interpolation of higher order on an arbitrary system of nodes are given. Based on this result conditions for convergence of Hermite interpolation and Hermite–Fejér-type interpolation on an arbitrary system of nodes as well as Grünwald type theorems are essentially simplified and improved. © 2000 Academic Press

1. INTRODUCTION

Let $n \in \mathbb{N}$ $(n \geqslant 2)$, $m_{kn} \in \mathbb{N}$ $(k=1, 2, ..., n,\ n=2, 3, ...)$, and

$$X := \{x_{1n}, x_{2n}, ..., x_{nn}\}, \qquad 1 \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant -1. \tag{1.1}$$

In what follows, m_{kn}, x_{kn}, ... will be denoted by m_k, x_k, ..., respectively. Throughout this paper let $N := N_n := \sum_{k=1}^{n} m_{kn} - 1$ and $m := \sup_{n \geqslant 2} \max_{1 \leqslant k \leqslant n} m_{kn} < +\infty$. Denote by $\mathbf{P}_N$ the set of polynomials of degree at most N and by A_{jk} the fundamental polynomials for Hermite interpolation, i.e., $A_{jk} \in \mathbf{P}_N$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \qquad p=0, 1, ..., m_q-1, \quad j=0, 1, ..., m_k-1,$$
$$q, k=1, 2, ..., n. \tag{1.2}$$

The Hermite interpolation of $f \in C^{m-1}[-1, 1]$ is given by

$$H_{nm}^{*}(f, x) = \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} f^{(j)}(x_k)\, A_{jk}(x)$$

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and Hermite–Fejér interpolation for $f \in C[-1, 1]$ is given by

$$H_{nm}(f, x) = \sum_{k=1}^{n} f(x_k)\, A_{0k}(x).$$

To give an explicit formula for A_{jk} set

$$L_k(x) = \prod_{q=1,\, q \neq k}^{n} \left(\frac{x - x_q}{x_k - x_q} \right)^{m_q}, \quad k = 1, 2, ..., n,$$

$$b_{vk} = \frac{1}{v!} [L_k(x)^{-1}]^{(v)}_{x = x_k}, \qquad v = 0, 1, ..., m_k - 1, \quad k = 1, 2, ..., n, \tag{1.3}$$

$$B_{jk}(x) = \sum_{v=0}^{m_k - j - 1} b_{vk}(x - x_k)^v, \qquad j = 0, 1, ..., m_k - 1, \quad k = 1, 2, ..., n. \tag{1.4}$$

Then by the same argument as in [10, Lemma 1] we have

$$A_{jk}(x) = \frac{1}{j!} (x - x_k)^j B_{jk}(x)\, L_k(x), \qquad j = 0, 1, ..., m_k - 1, \quad k = 1, 2, ..., n. \tag{1.5}$$

The most interesting special case is $m_k \equiv m$. In this case we have the simple formulas for $k = 1, 2, ..., n$

$$L_k(x) = \ell_k(x)^m,$$

where

$$\ell_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)}, \qquad \omega_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Although there have been many papers on Hermite interpolation of higher order (cf. [12] and its references), almost all of them discuss only interpolation based on the special system of nodes, say, zeros of Jacobi polynomials. However, only recently, Szabados [10] gives a very important result dealing with the Hermite interpolation of higher order on general nodes (see Lemma A below). It provides a deep estimation for the fundamental polynomials; based on this estimation a general Faber-type theorem (see Theorem B below) is proved and other applications are obtained [7–9, 13]; meanwhile its technique of proof is nice and suitable to other cases. The first aim of this paper is to give more general and stronger estimations of bounds for the fundamental functions of Hermite interpolation of higher order on an arbitrary system of nodes using many ideas of [10] in Section 2. This result will play a crucial role in the theory of Hermite interpolation and will make it possible to extend many important results previously obtained for Lagrange interpolation ($m=1$) and classical Hermite–Fejér interpolation ($m=2$) to Hermite interpolation of higher order ($m \geqslant 3$). As applications of this result, the second aim of this paper is to provide general and powerful criteria of convergence of Hermite interpolation and Hermite–Fejér-type interpolation of higher order on an arbitrary system of nodes in Sections 3 and 4, respectively. In the last section essentially simplified and improved Grünwald-type theorems are given.

2. BASIC THEOREMS

The proof of Theorem 2.1 below follows the line given by Szabados [10] but needs to use Birkhoff interpolation. For convenience we state some knowledge of Birkhoff interpolation [4, pp. 2–10]. A matrix $E=[e_{qp}]_{q=1,}^{n}\,{}_{p=0}^{N}$ is called a normal interpolation matrix if its elements e_{qp} are 0 or 1 and if the number of 1's in E is equal to $N+1$, $|E|=\sum e_{qp}=N+1$. Here we do not allow empty rows, i.e., in each q, $1 \leqslant q \leqslant n$, at least one e_{qp} is not zero. A Birkhoff interpolation problem E, X (with respect to $\mathbf{P}_N$) is, given a set of data y_{qp} (defined for $e_{qp}=1$), to determine a polynomial $P \in \mathbf{P}_N$ (if any) such that

$$P^{(p)}(x_q)=y_{pq}, \qquad e_{qp}=1, \qquad e_{qp} \in E. \tag{1.6}$$

The pair E, X is called regular if the system of Eqs. (1.6) has a unique solution $P=0$ for $y_{qp} \equiv 0$. A row q of the matrix E is said to be Hermitian if for some r, $e_{qp}=1$ for $p<r$ and $e_{qp}=0$ for $p \geqslant r$. A matrix E is said to be Hermitian if it has only Hermitian rows.

For normal matrices the condition

$$\sum_{p=0}^{s} \sum_{q=1}^{n} e_{qp} \geqslant s+1, \qquad s=0, 1, ..., N$$

is called the Pólya condition. A sequence of 1's of the qth row of E is supported if that (q, p) is the position of the first 1 of the sequence implies that there exist two 1's: $e_{q_1, p_1}=e_{q_2, p_2}=1$ with $q_1<q<q_2$, $p_1<p$, and $p_2<p$. Then we have

Theorem A [4, Theorem 1.5, p. 10]. *A normal interpolation matrix is regular for algebraic interpolation if its satisfies the Pólya condition and contains no odd supported sequences.*

In the following c, c_1, c_1^*, ... will stand for positive constants depending only on m, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas. Recently, Szabados proved an important result in which $\bar{d}_1 = x_1 - x_2$, $\bar{d}_n = x_{n-1} - x_n$, $\bar{d}_k = \max\{|x_k - x_{k-1}|, |x_k - x_{k+1}|\}$, $2 \leqslant k \leqslant n-1$ and $m_{kn} \equiv m$ means $m_{kn} = m$, $k = 1, 2, ..., n$, $n = 2, 3, ...$.

Lemma A [10, Lemma 3]. *Let $m_{kn} \equiv m$. If $m - j$ is odd then*

$$B_{jk}(x) \geqslant c_1^* \left(\frac{x - x_k}{\bar{d}_k} \right)^{m-j-1}, \qquad x \in \mathbb{R}, \quad 0 \leqslant j \leqslant m-1, \quad 1 \leqslant k \leqslant n. \tag{2.1}$$

This lemma plays an important role in proving the following general Faber-type theorem in which $\|\cdot\|$ denotes the uniform norm.

Theorem B [10, Theorem 1]. *Let $m_{kn} \equiv m$. Then*

$$\left\| \sum_{k=1}^{n} |A_{jk}| \right\| \geqslant \begin{cases} c_2^* n^{-j} \ln n, & \text{if} \quad m-j \text{ is odd}, \\ c_3^* n^{-j}, & \text{if} \quad m-j \text{ is even}, \end{cases} \qquad 0 \leqslant j \leqslant m-1. \tag{2.2}$$

Moreover, the order is the best possible and is attained by the Chebyshev nodes:

$$x_k = \cos \frac{2k-1}{2n} \pi, \qquad k = 1, 2, ..., n. \tag{2.3}$$

Meanwhile, Lemma A may be applicable to estimation of lower bounds of Lebesgue function-type sums [9, 13] and investigation of mean convergence for Hermite interpolation [8], as well as determination of asymptotic behavior for Cotes numbers of Gauss–Turán quadrature formulas [7].

To get further and more applications we need to extend and to strengthen this estimation. To this end let

$$d_k = \max\{|x_k - x_{k-1}|, |x_k - x_{k+1}|\}, \qquad k = 1, 2, ..., n,$$

and

$$I = \begin{cases} \mathbb{R}, & 2 \leqslant k \leqslant n-1, \\ (-\infty, 1], & k = 1, \\ [-1, +\infty), & k = n. \end{cases}$$

We give two lemmas before stating the main result.

LEMMA 2.1. *Let E be an $n\times(N+1)$ $(n\geqslant 4)$ Hermitian matrix with the lengths $m_1, m_2, \ldots, m_n$. Let k, $2\leqslant k\leqslant n-1$, be fixed and $0\leqslant j<i<m_k$, where both m_k-i and m_k-j are odd. Let a matrix E^* be obtained from E by omitting the two 1's in the positions (k, j) and (k, i). Assume that a polynomial G is annihilated by the pair E^*, X and satisfies*

$$\partial G=|E^*|-1. \tag{2.4}$$

Then

$$|G^{(j)}(x_k)|\leqslant\frac{j!}{i!}\,d_k^{i-j}\,|G^{(i)}(x_k)|. \tag{2.5}$$

Proof. We write the Taylor expression of G about $x=x_k$

$$G(x)=c_0(x-x_k)^j+c_1(x-x_k)^i+(x-x_k)^{m_k}\,D(x),\qquad D\in\mathbf{P}_N, \tag{2.6}$$

where

$$c_0=\frac{G^{(j)}(x_k)}{j!},\qquad c_1=\frac{G^{(i)}(x_k)}{i!}. \tag{2.7}$$

Thus

$$H(x):=\frac{G(x)}{c_0(x-x_k)^j}=1+\frac{c_1}{c_0}\,(x-x_k)^{i-j}+\frac{1}{c_0}\,(x-x_k)^{m_k-j}\,D(x) \tag{2.8}$$

and

$$H'(x)=(x-x_k)^{i-j-1}\,F(x), \tag{2.9}$$

where

$$F(x)=(i-j)\,\frac{c_1}{c_0}+(x-x_k)^{m_k-i}\,D_1(x),\qquad D_1\in\mathbf{P}_N. \tag{2.10}$$

Denote by ∂P the exact degree of $P\in\mathbf{P}_N$ and by $Z(P, \Delta)$ the number of zeros of P in Δ counting multiplicities. Let $\Delta_k=(x_{k+1}, x_{k-1})$ and let (a, b) be the largest open interval such that $(a, b)\supset\Delta_k$ and $Z(F, (a, b))=0$. We claim

$$Z(F, \Delta_k)=0 \tag{2.11}$$

and

$$Z(F', (a, b))\leqslant m_k-i. \tag{2.12}$$

In fact, by (2.8) we see that

$$H(x_q)=H'(x_q)=\cdots=H^{(m_q-1)}(x_q)=0, \qquad q\neq k, \quad 1\leqslant q\leqslant n, \tag{2.13}$$

and

$$\begin{aligned} H'(x_k)=\cdots&=H^{(i-j-1)}(x_k)=H^{(i-j+1)}(x_k)=\cdots \\ &=H^{(m_k-j-1)}(x_k)=0. \end{aligned} \tag{2.14}$$

By Rolle's theorem we obtain $r\geqslant n-3$ zeros $z_1, ..., z_r$ (outside Δ_k) of $H'(x)$ from $n-2$ intervals given by zeros $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n$ of $H(x)$:

$$H'(z_q)=0, \qquad q=1, ..., r. \tag{2.15}$$

Let E' be the interpolation matrix for H' corresponding to (2.13)–(2.15), i.e., $e_{qp}=1$ for $(H')^{(p)}(x_q)=0$ and $e_{qp}=0$ otherwise. Then H' is annihilated by the pair E', $X'=X\cup\{z_1, ..., z_r\}\backslash\{x_q : m_q=1\}$ (the nodes x_q in the last subset do not appear in (2.13)), which by Theorem A is regular (since m_k-i-1 is even). So we must have $|E'|\leqslant\partial H'\leqslant N-j-3$. But

$$|E'|=\sum_{q\neq k}(m_q-1)+(m_k-j-2)+r=N-j-n+r.$$

Hence $r\leqslant n-3$. Recalling $r\geqslant n-3$, we get

$$r=n-3 \tag{2.16}$$

and

$$|E'|=\partial H'=N-j-3. \tag{2.17}$$

Equation (2.11) follows from (2.17) and (2.9), for otherwise $H'=0$ would occur, a contradiction. By (2.9), (2.14), and (2.17) we also get

$$Z(F, \mathbb{R}\backslash(a, b))=|E'|-(m_k-j-2)=N-m_k-1$$

and hence $Z(F', \mathbb{R}\backslash(a, b))\geqslant N-m_k-3$. Since $\partial F'\leqslant N-i-3$, we have $Z(F', (a, b))\leqslant m_k-i$. This proves (2.11) and (2.12).

Noticing $n\geqslant 4$, by the definition of the interval (a, b) we conclude either $F(a)=0$ or $F(b)=0$. Assume without loss of generality that $F(b)=0$. Since by (2.9)

$$F'(x_k)=\cdots=F^{(m_k-i-1)}(x_k)=0, \tag{2.18}$$

according to (2.12) we can conclude that in the interval (a, b) the function F' has either a unique zero y of odd multiplicities or no zero of odd multiplicities. In the latter case we agree $y = -\infty$. Now put

$$\delta_k = \begin{cases} (x_{k+1}, x_k), & y \geqslant x_k, \\ (x_k, x_{k-1}), & y < x_k. \end{cases}$$

Then F is monotone in δ_k and hence by (2.10) and (2.18) we have

$$|F(x)| \leqslant |F(x_k)| = (i-j) \left| \frac{c_1}{c_0} \right|, \qquad x \in \delta_k.$$

Thus by (2.8), (2.9), and (2.7)

$$\begin{aligned} 1 &= \left| \int_{\delta_k} H'(x)\,dx \right| = \left| \int_{\delta_k} (x - x_k)^{i-j-1} F(x)\,dx \right| \\ &\leqslant (i-j) \left| \frac{c_1}{c_0} \right| \int_{\delta_k} |x - x_k|^{i-j-1}\,dx \\ &= \left| \frac{c_1}{c_0} \right| |\delta_k|^{i-j} \leqslant \left| \frac{c_1}{c_0} \right| d_k^{i-j} = \frac{j!\, d_k^{i-j}\, |G^{(i)}(x_k)|}{i!\, |G^{(j)}(x_k)|}, \end{aligned}$$

which is equivalent to (2.5). ▌

LEMMA 2.2. *Let k, $1 \leqslant k \leqslant n$, and*

$$a_{jk} = \sum_{i \neq k} \frac{m_i}{(x_i - x_k)^j}, \qquad j \geqslant 1. \tag{2.19}$$

Then

$$b_{jk} = \frac{1}{j} \sum_{i=1}^{j} a_{ik} b_{j-i,k}, \qquad j \geqslant 1. \tag{2.20}$$

Proof. Use the same argument as that of Lemma 2 of [10]. ▌

The first main result of this section is as follows.

THEOREM 2.1. *If for a fixed n, $m_k - j$ is odd and $j < i < m_k$, $1 \leqslant k \leqslant n$, then with $c = 1$*

$$B_{jk}(x) \geqslant c d_k^{j-i} |x - x_k|^{i-j} |B_{ik}(x)|, \qquad x \in I, \tag{2.21}$$

$$|A_{jk}(x)| \geqslant c \frac{i!}{j!} d_k^{j-i} |A_{ik}(x)|, \qquad x \in I, \tag{2.22}$$

and

$$b_{m_k-j-1,k} \geqslant c d_k^{j-i} |b_{m_k-i-1,k}|. \tag{2.23}$$

Moreover, both the order d_k^{j-i} *and the constant* $c=1$ *in* (2.21)–(2.23) *are the best possible.*

Proof. We begin by proving that

$$b_{m_k-j-1,k} > 0; \qquad B_{jk}(x) > 0, \quad x \in \mathbb{R}, \tag{2.24}$$

and

$$(-1)^\nu b_{\nu 1} > 0, \qquad \nu = 0, 1, ..., m_1 - 1;$$

$$b_{\nu n} > 0, \qquad \nu = 0, 1, ..., m_n - 1. \tag{2.25}$$

Let $E = E_{jk} = [e_{qp}]_{q=1,}^{n} {}_{p=0}^{N-1}$ be a normal matrix defined by $e_{qp} = 1$ for $A_{jk}^{(p)}(x_q) = 0$, $p \leqslant m_q - 1$, $q = 1, 2, ..., n$, and $e_{qp} = 0$ otherwise. Clearly, according to (1.2) A_{jk} is annihilated by the pair E, X. That is, $A_{jk}^{(p)}(x_q) = 0$, $e_{qp} = 1$, $e_{qp} \in E$. Since E satisfies the Pólya condition and contains no odd supported sequences ($m_k - j - 1$ is even), by Theorem A the pair E, X is regular. If $b_{m_k-j-1,k} = 0$ then $\partial A_{jk} \leqslant N-1$ and $A_{jk} = 0$, a contradiction. Moreover, suppose to the contrary that $B_{jk}(z) = 0$ for some $z \in \mathbb{R}$. If $z = x_t$ (of course $t \neq k$) then we add a 1 to the position (t, m_t) in E and put $X' = X$; if $z \notin X$ we add a new Lagrangian row $(1, 0, ..., 0)$ and put $X' = X \cup \{z\}$; let E' be obtained from E by the above process. Again, A_{jk} is annihilated by the pair E', X' and the pair E', X' is also regular. Thus it leads to a contradiction $A_{jk} = 0$. So (2.24) follows.

Since (2.24) implies (2.25) for the case when $m_k - \nu$ is odd, we have only to discuss the case when $m_k - \nu$ is even. By the same argument we also conclude that for $k=1$ and $k=n$ if $m_k - \nu$ is even then $b_{m_k-\nu-1,k} \neq 0$ and $B_{\nu k}(x)$ has exactly one zero which is in $(x_1, +\infty)$ for $k=1$ and in $(-\infty, x_n)$ for $k=n$, respectively. Since $B_{\nu k}(x_k) = 1$, we see that $b_{m_1-\nu-1,1} < 0$ and $b_{m_n-\nu-1,n} > 0$, which by (2.24) yields (2.25).

In the following proof we separate two cases.

Case 1. $m_k - i$ is odd. Denote

$$I' := \begin{cases} \mathbb{R}, & 2 \leqslant k \leqslant n-1, \\ (-\infty, x_1 + d_1], & k = 1, \\ [x_n - d_n, +\infty), & k = n. \end{cases}$$

To prove (2.21) put (since $x_1 = 1$ or $x_n = -1$ may occur, here we need to use I' instead of I)

$$c := c_{ijk} = \sup\{d : B_{jk}(x) \geqslant d(x - x_k)^{i-j} B_{ik}(x),\ x \in I'\}.$$

Of course $I \subset I'$,

$$C(x) := C_{jk}(x) = B_{jk}(x) - c(x - x_k)^{i-j} B_{ik}(x) \geqslant 0, \qquad x \in I',$$

and by (1.5)

$$G(x) := C(x)\, L_k(x) \frac{(x - x_k)^j}{j!} = A_{jk}(x) - \frac{ci!}{j!} A_{ik}(x). \tag{2.26}$$

Claim 1. $Z(C) \leqslant 2$.

Suppose to the contrary that the polynomial C has three zeros, say, $\alpha_1 = \alpha_2 > \alpha_3 : C(\alpha_1) = C'(\alpha_1) = C(\alpha_3) = 0$. Hence $G(\alpha_1) = G'(\alpha_1) = G(\alpha_3) = 0$. Then according to Rolle's theorem $Z(G^{(i+1)}) = |E| - 1 - 2 + 3 - (i+1) = |E| - i - 1$ and $\partial G^{(i+1)} = \partial G - (i+1) = |E| - i - 2 < Z(G^{(i+1)})$, a contradiction. This proves Claim 1.

Claim 2. One and only one of the following three cases may occur:

Case A. The polynomial C with $\partial C = \partial B_{jk}$ has two zeros $\alpha_1 \leqslant \alpha_2$, which satisfy $I' \cap \{\alpha_1, \alpha_2\} \neq \emptyset$ and

$$\begin{cases} C(\alpha_1) = C'(\alpha_1) = 0, & \alpha_1 = \alpha_2, \\ C(\alpha_1) = C(\alpha_2) = 0, & \alpha_1 \neq \alpha_2; \end{cases} \tag{2.27}$$

Besides, $\alpha_1 > x_1$ for $k = 1$ and $\alpha_2 < x_n$ for $k = n$.

Case B. The polynomial C with $\partial C = \partial B_{jk} - 1$ has a unique simple zero $\alpha_1 = x_1 + d_1$ or $\alpha_1 = x_n - d_n$;

Case C. The polynomial C with $\partial C = \partial B_{jk} - 2$ has no zero and $2 \leqslant k \leqslant n-1$.

In fact, if $Z(C) = 2$, denoting the zeros of C by α_1, α_2, $\alpha_1 \geqslant \alpha_2$, then $I' \cap \{\alpha_1, \alpha_2\} \neq \emptyset$ and (2.27) holds. By the same argument as that of Claim 1 we can obtain the last conclusion. This proves Case A. If $Z(C) = 1$, then $Z(C, I' \backslash (I')^\circ) = Z(C, I') - Z(C, (I')^\circ) = 1$ ($(I')^\circ$ denotes the set of interior points of I') and, recalling that ∂B_{jk} is even, $\partial C \leqslant \partial B_{jk} - 1$. On the other hand, we must have $\partial C \geqslant \partial B_{jk} - 1$, for otherwise by a similar argument as that of Claim 1 it would lead to a contradiction. This proves Case B. If $Z(C) = 0$, then $\partial C \leqslant \partial B_{jk} - 2$ and we also must have $\partial C \geqslant \partial B_{jk} - 2$ by the same argument as above. Furthermore we have $2 \leqslant k \leqslant n-1$, for otherwise it again leads to a contradiction. This proves Claim 2.

According to Claim 2 for Case A

$$\begin{cases} G(\alpha_1)=G'(\alpha_1)=0, & \alpha_1=\alpha_2, \\ G(\alpha_1)=G(\alpha_2)=0, & \alpha_1\neq\alpha_2, \end{cases} \tag{2.28}$$

and for Case B

$$G(\alpha_1)=0, \qquad \alpha_1=x_1+d_1 \quad \text{or} \quad \alpha_1=x_n-d_n. \tag{2.29}$$

Let E_{kji} be obtained from E by omitting its two 1's in the positions (k, j) and (k, i). Let $E^*=E_{kji}$ for Case C; let E^* be obtained from E_{kji} by adding two 1's corresponding to the zeros in (2.28) for Case A with $\alpha_1\neq\alpha_2$ or a 1 corresponding to the zero in (2.29) for Case B, either in a new Lagrangian row $(1, 0, ..., 0)$ or as an additional 1 in an old row, just after the sequence; otherwise we add a new row $(1, 1, 0, ..., 0)$ if $\alpha_1=\alpha_2\notin X$ and two additional 1's in the μth row if $\alpha_1=\alpha_2=x_\mu$, just after the sequence. Clearly, the pair E^*, $X^*=X\cup\{\alpha_1, \alpha_2\}:=\{x_1^*, ..., x_n^*\}$ with $x_1^*>\cdots>x_n^*$ annihilates G. If we put $\rho=2, 1, 0$ for Cases A, B, C, respectively, then $\partial G=|E|-1-(2-\rho)=|E|+\rho-3$ and $|E^*|=|E|+\rho-2$. Thus $\partial G=|E^*|-1$. Then we can apply Lemma 2.1 to the pair E^*, X^* and the polynomial G. Since $X^*=X\cup\{\alpha_1, \alpha_2\}=\{x_1^*, ..., x_n^*\}$, there is an index k^* such that $x_{k^*}^*=x_k$. We get

$$\begin{aligned} 1=G^{(j)}(x_k)=G^{(j)}(x_{k^*}^*) &\leqslant \frac{j!(d_{k^*}^*)^{i-j}}{i!}|G^{(i)}(x_{k^*}^*)| \\ &= \frac{j!(d_{k^*}^*)^{i-j}}{i!}|G^{(i)}(x_k)| = \frac{j!(d_{k^*}^*)^{i-j}}{i!}\cdot\frac{ci!}{j!}=c(d_{k^*}^*)^{i-j}, \end{aligned} \tag{2.30}$$

where

$$d_{k^*}^*=\max\{|x_{k^*}^*-x_{k^*-1}^*|, |x_{k^*}^*-x_{k^*+1}^*|\}.$$

If we can show

$$d_{k^*}^*\leqslant d_k, \qquad 1\leqslant k\leqslant n, \tag{2.31}$$

then (2.30) implies $c\geqslant(d_{k^*}^*)^{j-i}\geqslant d_k^{j-i}$, which proves (2.21). Let us show (2.31). For $2\leqslant k\leqslant n-1$ since $X^*=X\cup\{\alpha_1, \alpha_2\}$ with $\alpha_1=\alpha_2$, we see that $d_{k^*}^*=d_k$ if $\alpha_1\notin[x_{k+1}, x_{k-1}]$ and $d_{k^*}^*\leqslant d_k$ otherwise. For $k=1$ only Cases A and B can occur. In this case Claim 2 says $\alpha_1\in(x_1, x_1+d_1]$. Hence

$$d_{1^*}^*=\max\{|x_1-x_2|, |x_1-\alpha_1|\}\leqslant\max\{|x_1-x_2|, d_1\}=d_1.$$

Similarly, we can show $d_{n^*}^*\leqslant d_n$.

Case 2. $m_k - i$ is even. In this case it suffices to show (2.21) for $i = j+1$, since validity of (2.21) for $i = j+1$ by the conclusion of Case 1 implies validity of (2.21) for $i \geqslant j+3$:

$$B_{jk}(x) \geqslant \left(\frac{x - x_k}{d_k}\right)^{i-j-1} B_{i-1,k}(x) \geqslant \left|\frac{x - x_k}{d_k}\right|^{i-j} |B_{ik}(x)|.$$

Now let us prove (2.21) for $i = j+1$. To this end put

$$L_k^*(x) = L_k(x) \left(\frac{x - x_r}{x_k - x_r}\right)^{-1}, \qquad r \neq k.$$

Thus

$$b_{\nu k}^* = \frac{1}{\nu!} [L_k^*(x)^{-1}]_{x = x_k}^{(\nu)} = b_{\nu k} + \frac{1}{x_k - x_r} b_{\nu-1,k}, \qquad \nu \geqslant 1. \tag{2.32}$$

Then by (2.24)

$$B_{jk}^*(x) = \sum_{\nu=0}^{m_k - j - 1} b_{\nu k}^*(x - x_k)^\nu = B_{jk}(x) + \frac{x - x_k}{x_k - x_r} B_{j+1,k}(x) > 0. \tag{2.33}$$

For $2 \leqslant k \leqslant n-1$ substituting $r = k \pm 1$ into (2.33) yields (2.21) with $c = 1$.

Let $k = 1$. Then (2.33) with $r = 2$ gives only

$$B_{j1}(x) > -\frac{x - x_1}{x_1 - x_2} B_{j+1,1}(x)$$

and it suffices to show

$$B_{j1}(x) > \frac{x - x_1}{x_1 - x_2} B_{j+1,1}(x), \qquad x \in I_n', \tag{2.32}$$

since it coupled with the previous inequality implies

$$B_{j1}(x) > \left|\frac{x - x_1}{x_1 - x_2} B_{j+1,1}(x)\right| \geqslant \left|\frac{x - x_1}{d_1} B_{j+1,1}(x)\right|, \qquad x \in I_n'.$$

At the beginning of the proof we proved that $B_{j+1,1}(x)$ has exactly one zero, say ξ, which must lie in $(x_1, +\infty)$. Since $B_{j+1,1}(x_1) = 1$, we see that

$B_{j+1,1}(x) \geqslant 0$ for $x \leqslant \xi$ and $B_{j+1,1}(x) < 0$ for $x > \xi$. Then $(x-x_1)\, B_{j+1,1}(x) \leqslant 0$ for $x \leqslant x_1$ or $x \geqslant \xi$ and (2.32) holds for these x. Now for $x \in (x_1, \min\{\xi, x_1 + \bar{d}_1\})$ by (2.25) we have

$$B_{j1}(x) = b_{m_1-j-1,1}(x-x_1)^{m_1-j-1} + B_{j+1,1}(x)$$

$$\geqslant B_{j+1,1}(x) \geqslant \frac{x-x_1}{d_1}\, B_{j+1,1}(x).$$

This proves (2.32) and (2.21) for $k=1$. Similarly we can prove (2.21) for $k=n$.

This completes the proof of (2.21).

By (1.5) and (2.21)

$$|A_{jk}(x)| = \left|\frac{1}{j!}(x-x_k)^j\, B_{jk}(x)\, L_k(x)\right|$$

$$\geqslant \frac{i!}{j!}\, d_k^{j-i} \left|\frac{1}{i!}(x-x_k)^i\, B_{ik}(x)\, L_k(x)\right|$$

$$= \frac{i!}{j!}\, d_k^{j-i}\, |A_{ik}(x)|.$$

This proves (2.22). Comparing the leading coefficients of both the sides of (2.21) yields (2.23).

To prove the last conclusion it is enough to show the last conclusion for (2.23), because (2.21) is equivalent to (2.22) and (2.21) implies (2.23).

To show that the order is the best possible let $m_{kn} \equiv m$ and let the system X of nodes be as follows, in which k, $1 \leqslant k \leqslant n$, is fixed:

$$x_{k-1} = \frac{1}{2n^2}, \qquad x_k = 0, \quad x_{k+1} = -\frac{1}{n^2}, \quad |x_i| \geqslant \frac{1}{2}, \quad |i-k| \geqslant 2.$$

Then $d_k = 1/n^2$,

$$\left|\sum_{|i-k| \geqslant 2} \frac{m}{x_i^j}\right| \leqslant m 2^j (n-2),$$

and

$$a_{jk} = \sum_{i \neq k} \frac{m}{(x_i - x_k)^j} = \sum_{i \neq k} \frac{m}{x_i^j}$$

$$= m(2n^2)^j + m(-n^2)^j + \sum_{|i-k| \geqslant 2} \frac{m}{x_i^j} \sim n^{2j} \sim d_k^{-j}.$$

Thus by (2.20) we obtain

$$b_{jk} \sim d_k^{-j} \sim d_k^{i-j} b_{ik}, \qquad i > j \geqslant 0.$$

This proves that the order of (2.23) is the best possible.

To show that the constant $c = 1$ of (2.23) is the best possible. Let us consider the system X of nodes $(n = 2p-1,\ p \geqslant 3)$:

$$x_p = 0, \qquad x_{p-1} = \frac{1}{p}, \qquad x_{p-2} \geqslant \frac{1}{2},$$
$$x_{p-k} = -x_{p+k}, \qquad k = 1, 2, ..., p-1.$$

Besides we choose

$$m_k = \begin{cases} 3, & k = p, \\ 1, & k \neq p. \end{cases}$$

Then by (2.19) and (2.20) $a_{1p} = 0$, $b_{2p} = (a_{1p}^2 + a_{2p})/2 = a_{2p}/2$, and by (2.23)

$$c \leqslant d_p^2 b_{2p} = \frac{x_{p-1}^2}{2} \sum_{i \neq p} \frac{m_i}{(x_i - x_p)^2} = x_{p-1}^2 \sum_{i=1}^{p-1} \frac{1}{x_i^2}$$
$$= 1 + x_{p-1}^2 \sum_{i=1}^{p-2} \frac{1}{x_i^2} \leqslant 1 + \frac{4(p-2)}{p^2} \to 1,$$

as $p \to \infty$. This proves our conclusion. ∎

The second main result in this section is

THEOREM 2.2. *Let $m_{kn} \equiv m \geqslant 3$. Then*

$$\begin{aligned} &B_{m-3,k}(x) \geqslant \tfrac{1}{2} B_{m-1,k}(x) = \tfrac{1}{2}, \\ &B_{m-3,k}(x) \geqslant |B_{m-2,k}(x)|, \qquad x \in \mathbb{R}, \quad 1 \leqslant k \leqslant n, \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} &|A_{m-1,k}(x)| \leqslant |(x-x_k)^2 A_{m-3,k}(x)|, \\ &|A_{m-2,k}(x)| \leqslant |(x-x_k)\, A_{m-3,k}(x)|, \qquad x \in \mathbb{R}, \quad 1 \leqslant k \leqslant n. \end{aligned} \tag{2.34}$$

Proof. By (1.3) we have

$$\begin{cases} b_{0k} = 1, \\ b_{1k} = -m\ell'_k(x_k), \\ b_{2k} = \tfrac{1}{2} m(m+1)\, \ell'_k(x_k)^2 - \tfrac{1}{2} m \ell''_k(x_k). \end{cases}$$

By a well-known result (cf. [11, p. 976])

$$\ell_k'(x_k)^2 - \ell_k''(x_k) = \sum_{i \neq k} \frac{1}{(x_k - x_i)^2} > 0 \tag{2.35}$$

we have

$$b_{2k} \geqslant \tfrac{1}{2} m^2 \ell_k'(x_k)^2.$$

Using a simple symbol $y := \ell_k'(x_k)(x - x_k)$ by (1.4) we get

$$\begin{cases} B_{m-1,k}(x) = 1, \\ B_{m-2,k}(x) = -my + 1, \\ B_{m-3,k}(x) \geqslant \tfrac{1}{2} m^2 y^2 - my + 1. \end{cases}$$

Hence

$$B_{m-3,k}(x) - \tfrac{1}{2} \geqslant \tfrac{1}{2} m^2 y^2 - my + \tfrac{1}{2} = \tfrac{1}{2}(my - 1)^2 \geqslant 0,$$

$$B_{m-3,k}(x) - B_{m-2,k}(x) \geqslant \tfrac{1}{2} m^2 y^2 \geqslant 0,$$

$$B_{m-3,k}(x) + B_{m-2,k}(x) \geqslant \tfrac{1}{2} m^2 y^2 - 2my + 2 = \tfrac{1}{2}(my - 2)^2 \geqslant 0,$$

which is equivalent to (2.33). Equation (2.34) follows directly from (2.33) and (1.5). ▮

LEMMA 2.3. *Let* $m_{kn} \equiv m$. *Then*

$$\sum_{k=1}^{n} (x - x_k)^i A_{0k}(x) = i! \sum_{j=1}^{m-1} \frac{(-1)^{j+1}}{(i-j)!} \sum_{k=1}^{n} (x - x_k)^{i-j} A_{jk}(x), \qquad 1 \leqslant i \leqslant nm - 1. \tag{2.36}$$

(*We agree to replace* $1/r!$ *by* 0 *if* $r < 0$.)

Proof. We have the identity

$$x^p = \sum_{j=0}^{m-1} \sum_{k=1}^{n} \frac{p!}{(p-j)!} x_k^{p-j} A_{jk}(x), \qquad 0 \leqslant p \leqslant mn - 1. \tag{2.37}$$

Using (2.37) we obtain

$$\begin{aligned}
&\sum_{j=0}^{m-1} \frac{(-1)^j}{(i-j)!} \sum_{k=1}^{n} (x-x_k)^{i-j} A_{jk}(x) \\
&\quad = \sum_{j=0}^{m-1} \frac{(-1)^j}{(i-j)!} \sum_{k=1}^{n} A_{jk}(x) \sum_{p=0}^{i-j} (-1)^p \binom{i-j}{p} x^{i-j-p} x_k^p \\
&\quad = \sum_{j=0}^{m-1} \sum_{k=1}^{n} A_{jk}(x) \sum_{p=0}^{i-j} \frac{(-1)^{p+j}}{p!(i-j-p)!} x^{i-j-p} x_k^p \\
&\quad = \sum_{j=0}^{m-1} \sum_{k=1}^{n} A_{jk}(x) \sum_{p=0}^{i} \frac{(-1)^p}{(p-j)!\,(i-p)!} x^{i-p} x_k^{p-j} \\
&\quad = \sum_{p=0}^{i} \frac{(-1)^p x^{i-p}}{p!(i-p)!} \sum_{j=0}^{m-1} \sum_{k=1}^{n} \frac{p!}{(p-j)!} x_k^{p-j} A_{jk}(x) \\
&\quad = \frac{x^i}{i!} \sum_{p=0}^{i} (-1)^p \binom{i}{p} = 0.
\end{aligned}$$

This is equivalent to (2.36). ∎

The last main result in this section is

THEOREM 2.3. *Let $m_{kn} \equiv m$ be even. Then*

$$\sum_{k=1}^{n} (x-x_k)\, A_{1k}(x) = \sum_{k=1}^{n} |(x-x_k)\, A_{1k}(x)| \leqslant \sum_{k=1}^{n} (x-x_k)^2 A_{0k}(x), \qquad x \in \mathbb{R}. \tag{2.38}$$

Proof. By (2.36), (1.5), and (2.24)

$$\begin{aligned}
\sum_{k=1}^{n} (x-x_k)^2 A_{0k}(x) &= 2 \sum_{k=1}^{n} [(x-x_k)\, A_{1k}(x) - A_{2k}(x)] \\
&= \sum_{k=1}^{n} (x-x_k)^2 \ell_k(x)^m [2B_{1k}(x) - B_{2k}(x)] \\
&\geqslant \sum_{k=1}^{n} (x-x_k)^2 \ell_k(x)^m B_{1k}(x) \\
&= \sum_{k=1}^{n} (x-x_k)\, A_{1k}(x) = \sum_{k=1}^{n} |(x-x_k)\, A_{1k}(x)|.
\end{aligned}$$

Here we use a fact

$$B_{1k}(x)-B_{2k}(x)=b_{m-2,k}(x-x_k)^{m-2}\geqslant 0, \qquad x\in\mathbb{R}, \quad 1\leqslant k\leqslant n,$$

which follows from (2.24). ∎

3. CONVERGENCE OF HERMITE INTERPOLATION

Let

$$\|f\|^* := \max_{0\leqslant j\leqslant m-1} \|f^{(j)}\|, \qquad f\in C^{m-1}[-1,1],$$

$$\|H^*_{nm}\| := \sup_{\|f\|^*\leqslant 1} \|H^*_{nm}(f)\|,$$

$$\|H_{nm}\| := \sup_{\|f\|\leqslant 1} \|H_{nm}(f)\|.$$

It is well known that

$$\|H_{nm}\| = \left\| \sum_{k=1}^{n} |A_{0k}| \right\|.$$

The first main result in this section is the following

THEOREM 3.1. *Let $m_{kn}\equiv m\neq 2$. Then*

$$\|H^*_{nm}\| \leqslant c\, \|H_{nm}\|. \tag{3.1}$$

Proof. First let m be even. Then

$$\|H^*_{nm}\| \leqslant \sum_{j=0}^{m-1} \sup_{\|f\|^*\leqslant 1} \left\| \sum_{k=1}^{n} f^{(j)}(x_k)\, A_{jk}(x) \right\| := \sum_{j=0}^{m-1} S_j.$$

We separate the cases when $j=0$, $1\leqslant j\leqslant m-2$, and $j=m-1$. Clearly, $S_0\leqslant \|H_{nm}\|$. For $1\leqslant j\leqslant m-2$ by the mean value theorem for the derivative

$$\begin{aligned} S_j &= \sup_{\|f\|^*\leqslant 1} \left\| \sum_{k=1}^{n} \{f^{(j)}(x)-[f^{(j)}(x)-f^{(j)}(x_k)]\}\, A_{jk}(x) \right\| \\ &= \sup_{\|f\|^*\leqslant 1} \left\| \sum_{k=1}^{n} \{f^{(j)}(x)-f^{(j+1)}(\xi_k)(x-x_k)\}\, A_{jk}(x) \right\| \\ &\leqslant \left\| \sum_{k=1}^{n} A_{jk} \right\| + \left\| \sum_{k=1}^{n} |(x-x_k)\, A_{jk}(x)| \right\|. \end{aligned}$$

But by (2.36)

$$\left\| \sum_{k=1}^{n} A_{jk} \right\| = \left\| \sum_{i=0}^{j-1} \frac{(-1)^i}{(j-i)!} \sum_{k=1}^{n} (x-x_k)^{j-i} A_{ik}(x) \right\|.$$

We have that

$$\left\| \sum_{k=1}^{n} |(x-x_k)^j A_{0k}(x)| \right\| \leqslant 2^j \, \|H_{nm}\|$$

and by (2.22), using the inequalities $|x-x_k| \leqslant 2$ and $d_k \leqslant 2$,

$$\sum_{k=1}^{n} |(x-x_k)^{j-i} A_{ik}(x)| \leqslant 2^{j-2} \sum_{k=1}^{n} (x-x_k) \, A_{1k}(x), \qquad 1 \leqslant i \leqslant j-1.$$

Thus

$$S_j \leqslant 2^j \, \|H_{nm}\| + c \left\| \sum_{k=1}^{n} (x-x_k) \, A_{1k}(x) \right\|.$$

As for $j=m-1$ by (2.34) and (2.22)

$$S_{m-1} \leqslant \left\| \sum_{k=1}^{n} |A_{m-1,k}(x)| \right\| \leqslant \left\| \sum_{k=1}^{n} (x-x_k)^2 \, A_{m-3,k}(x) \right\|$$
$$\leqslant 2^{m-3} \left\| \sum_{k=1}^{n} (x-x_k) \, A_{1k}(x) \right\|.$$

At last by (2.38)

$$\|H^*_{nm}\| \leqslant \sum_{j=0}^{m-2} 2^j \, \|H_{nm}\| + c \left\| \sum_{k=1}^{n} (x-x_k) \, A_{1k}(x) \right\| \leqslant c \, \|H_{nm}\|.$$

For odd m it suffices to apply (2.22). ∎

Since H^*_{nm} is a linear operator from $C^{m-1}[-1, 1]$ to $\mathbf{P}_N$, as a direct consequence of Theorem 3.1 by the Banach–Steinhaus theorem we state the second main result in this section.

THEOREM 3.2. *Let $m_{kn} \equiv m \geqslant 4$ be even. If*

$$\|H_{nm}\| = \left\| \sum_{k=1}^{n} |A_{0k}| \right\| = \bigcirc(1), \tag{3.2}$$

then

$$\lim_{n\to\infty} \|H^*_{nm}(f)-f\|=0 \tag{3.3}$$

holds for every $f\in C^{m-1}[-1,1]$.

This result essentially simplifies to the following

THEOREM C [6, Lemma 3]. *Let* $m_{kn}\equiv m$ *be even. If*

$$\left\|\sum_{j=0}^{m-1}\sum_{k=1}^{n}|A_{jk}|\right\|=\bigcirc(1),$$

then (3.3) *holds for all* $f\in C^{m-1}[-1,1]$.

It is still open whether or not Theorems 3.1 and 3.2 remain true for $m_{kn}\equiv 2$. Until now we have only found two possible answers to this problem. The first is the following, which may be shown by the same argument as that of Theorems 3.1 and 3.2.

THEOREM 3.3. *Let* $m_{kn}\equiv 2$. *Then for* $f\in C^2[-1,1]$

$$\sup_{\|f\|\leqslant 1,\ \|f'\|\leqslant 1,\ \|f''\|\leqslant 1} \|H^*_{n2}(f)\|\leqslant c_3\,\|H_{n2}\|. \tag{3.4}$$

Moreover, if

$$\|H_{n2}\|=\left\|\sum_{k=1}^{n}|A_{0k}|\right\|=\bigcirc(1), \tag{3.5}$$

then

$$\lim_{n\to\infty} \|H^*_{n2}(f)-f\|=0 \tag{3.6}$$

holds for every $f\in C^2[-1,1]$.

To state the second answer we first formulate a general statement, which is an estimation of the lower bound of $\|H^*_{nm}\|$.

THEOREM 3.4. *Let* $m_{kn}\equiv m$. *Then*

$$\|H^*_{nm}\|\geqslant c\left\|\sum_{k=1}^{n}|A_{m-1,k}|\right\|. \tag{3.7}$$

Proof. To prove (3.7) we need a result which may be deduced by the same argument as that of [8, Lemma 2.5]:

LEMMA 3.1. *Let* $r \geqslant 0$, $n \in \mathbb{N}$, *and* $|\varepsilon_k| \leqslant 1$, $k = 1, 2, ..., n$. *Then there exists a function* $f \in C^r[-1, 1]$ *such that*

$$\begin{cases} f^{(r)}(x_k) = \varepsilon_k, & k = 1, 2, ..., n, \\ f^{(j)}(x_k) = 0, & k = 1, 2, ..., n; \quad j = 0, 1, ..., r-1, \\ \|f^{(j)}\| \leqslant \dfrac{2^r[(2r+1)!]^2}{r!\,[(r+1)!]^2}, & j = 0, 1, ..., r. \end{cases} \tag{3.8}$$

Now to apply this lemma let $r = m - 1$,

$$\sum_{k=1}^{n} |A_{m-1,k}(\xi)| = \left\| \sum_{k=1}^{n} |A_{m-1,k}| \right\|, \qquad \xi \in [-1, 1],$$

and $\varepsilon_k = \operatorname{sgn} A_{m-1,k}(\xi)$, $k = 1, 2, ..., n$. For the function f given by Lemma 3.1 according to (3.8)

$$\|H^*_{nm}\| \geqslant (\|f\|^*)^{-1} \|H^*_{nm}(f)\| \geqslant c\, \|H^*_{nm}(f)\| = c \left\| \sum_{k=1}^{n} |A_{m-1,k}| \right\|,$$

where

$$c = \frac{(m-1)!\,(m!)^2}{2^{m-1}[(2m-1)!]^2}. \quad ∎$$

Now we are able to state the second answer.

THEOREM 3.5. *Let* $m_{kn} \equiv 2$. *Then*

$$c_4 \left\| \sum_{k=1}^{n} |A_{1k}| \right\| \leqslant \|H^*_{n2}\| \leqslant \|H_{n2}\| + \left\| \sum_{k=1}^{n} |A_{1k}| \right\|. \tag{3.9}$$

Moreover,

$$\|H^*_{n2}\| \leqslant c_2\, \|H_{n2}\| \tag{3.10}$$

if and only if

$$\left\| \sum_{k=1}^{n} |A_{1k}| \right\| \leqslant c_7\, \|H_{n2}\|. \tag{3.11}$$

Proof. Equation (3.9) follows from (3.7) and the definition of $\|H^*_{n2}\|$. The equivalence of (3.10) and (3.11) follows from (3.9). ∎

4. CONVERGENCE OF HERMITE–FEJÉR-TYPE INTERPOLATION

Put

$$R_{nm}(f, x) := |H_{nm}(f, x) - f(x)|,$$

$$r_{nm}(x) := R_{nm}(f_1, x) + R_{nm}(f_2, x), \qquad f_i(x) := x^i, \quad i = 1, 2.$$

LEMMA 4.1. *Let* $P_k \in \mathbf{P}_n$, $k = 1, 2, ..., M$, *and* $1 \geqslant y_1 > y_2 > \cdots > y_M \geqslant -1$. *If*

$$\left\| \sum_{k=1}^{M} |(x - y_k) P_k(x)| \right\| = \mu_n \tag{4.1}$$

and

$$\sum_{k=1}^{M} |P_k(y_j)| \leqslant \nu_n, \qquad j = 1, 2, ..., M, \tag{4.2}$$

then

$$\left\| \sum_{k=1}^{M} |P_k| \right\| \leqslant 2(n^2 \mu_n + \nu_n). \tag{4.3}$$

In particular, if $M = 1$ *and* $P_1(y_1) = 0$, $|y_1| < 1$, *then*

$$\|P_1\| \leqslant \frac{4n\mu_n}{(1 - y_1^2)^{1/2}}.$$

Proof. Let

$$\sum_{k=1}^{M} |P_k(\xi)| = w_n := \left\| \sum_{k=1}^{m} |P_k| \right\|, \qquad \xi \in [-1, 1].$$

If $|\xi - y_k| \geqslant 1/(2n^2)$ holds for every k then

$$\mu_n \geqslant \sum_{k=1}^{M} |(\xi - y_k) P_k(\xi)| \geqslant \frac{w_n}{2n^2},$$

which implies (4.3).

If $|\xi - y_j| < 1/(2n^2)$ holds for some j then with the notation

$$P(x) = \sum_{k=1}^{M} [\operatorname{sgn} P_k(\xi)]\, P_k(x)$$

by the mean value theorem for the derivative and the Markov inequality

$$w_n - v_n \leqslant P(\xi) - P(y_j) = P'(\eta)(\xi - y_j)$$
$$\leqslant n^2 \|P\| \, |\xi - y_j| \leqslant n^2 w_n \cdot \frac{1}{2n^2} = \frac{1}{2} w_n.$$

Hence $w_n \leqslant 2v_n$, which implies (4.3).

To prove the second conclusion let $y_1 = \cos\theta$ and $\xi = \cos\tau$ and choose $t = \theta + \varepsilon\pi/(2n)$, where $\varepsilon = \operatorname{sgn}(\tau - \theta)$. It is enough to establish

$$|\xi - y_1| \geqslant \frac{\sin\theta}{4n}, \tag{4.4}$$

because by (4.1) and (4.4)

$$\mu_n \geqslant |(\xi - y_1)\, P_1(\xi)| \geqslant \frac{\sin\theta}{4n} \|P_1\| = \frac{(1 - y_1^2)^{1/2}}{4n} \|P_1\|.$$

To this end we need a theorem on Riesz [5] which says that if $P \in \mathbf{P}_n$ attains its absolute maximum in $[-1, 1]$ at $y = \cos\alpha$ then $|\alpha - \theta_k| \geqslant \pi/(2n)$ $(k = 1, 2, ..., n)$, where $z_k = \cos\theta_k$ $(k = 1, 2, ..., n)$ denote the roots of $P(x)$. Thus in the present case $|\tau - \theta| \geqslant \pi/(2n)$ and hence

$$|\xi - y_1| \geqslant |\cos t - \cos\theta| = \left| 2 \sin\frac{t+\theta}{2} \sin\frac{t-\theta}{2} \right|$$
$$= \left| 2 \sin\left(\theta + \frac{\varepsilon\pi}{4n}\right) \sin\frac{\pi}{4n} \right| \geqslant \left| \frac{1}{n} \sin\left(\theta + \frac{\varepsilon\pi}{4n}\right) \right|$$
$$= \frac{1}{n} \left| \sin\theta \cos\frac{\pi}{4n} + \varepsilon \cos\theta \sin\frac{\pi}{4n} \right|$$
$$\geqslant \frac{1}{n} \left[\frac{2^{1/2}}{2} \sin\theta - \sin\frac{\pi}{4n} \right] \geqslant \frac{1}{n} \left[\frac{2^{1/2}}{2} \sin\theta - \frac{\pi}{4n} \right].$$

If

$$\sin\theta \geqslant \frac{\pi}{4n\left(\dfrac{2^{1/2}}{2} - \dfrac{1}{4}\right)},$$

i.e.,

$$\frac{2^{1/2}}{2} \sin\theta - \frac{\pi}{4n} \geqslant \frac{1}{4} \sin\theta,$$

then (4.4) follows.

If

$$\sin\theta < \frac{\pi}{4n\left(\frac{2^{1/2}}{2} - \frac{1}{4}\right)},$$

then in this case $\sin\theta \leqslant 2/n$ and hence

$$|\xi - y_1| \geqslant |\cos t - \cos\theta| \geqslant |1 - \cos(t-\theta)|$$
$$\geqslant \left|1 - \cos\frac{\pi}{2n}\right| = 2\sin^2\frac{\pi}{4n} \geqslant \frac{1}{2n^2} \geqslant \frac{\sin\theta}{4n}. \quad \blacksquare$$

COROLLARY 4.1. *Let* $m_{kn} \equiv m$. *If*

$$\left\| \sum_{k=1}^{n} (x - x_k)\, A_{1k}(x) \right\| = \mu_n, \tag{4.5}$$

then

$$\left\| \sum_{k=1}^{n} |A_{1k}| \right\| \leqslant 2m^2n^2\mu_n. \tag{4.6}$$

Proof. Since

$$\sum_{k=1}^{n} |A_{1k}(x_j)| = 0, \qquad j = 1, 2, ..., n,$$

(4.6) directly follows from Lemma 4.1. $\blacksquare$

COROLLARY 4.2. *Let* $m_{kn} \equiv m$ *be even. If*

$$\left\| \sum_{k=1}^{n} |A_{0k}| \right\| = \mu_n, \tag{4.7}$$

then

$$\left\| \sum_{k=1}^{n} |A_{1k}| \right\| \leqslant 8m^2n^2\mu_n. \tag{4.8}$$

Proof. By (2.38) and (4.7) we have

$$\sum_{k=1}^{n} (x - x_k)\, A_{1k}(x) \leqslant 4\mu_n,$$

which according to Corollary 4.1 gives (4.8). $\blacksquare$

The main result in this section is the following

THEOREM 4.1. *Let $m_{kn} \equiv m$ be an even integer. Then for any $P \in \mathbf{P}_{mn-1}$*

$$R_{nm}(P, x) \leqslant c\, \|P\|^* \left\{ r_{nm}(x) + \frac{\|r_{nm}\| \ln^{10}[n(1+\|r_{nm}\|)]}{n} \right\}. \tag{4.9}$$

Furthermore, if

$$\lim_{n \to \infty} \|H_{nm}(f) - f\| = 0 \tag{4.10}$$

holds for $f = f_i$, $i = 1, 2$, then (4.10) *holds for every polynomial f.*

Proof. First let us prove two claims.

Claim 1. We have the inequality

$$\sum_{k=1}^{n} (x - x_k)\, A_{1k}(x) \leqslant 2r_{nm}(x), \qquad x \in [-1, 1]. \tag{4.11}$$

In fact,

$$\begin{aligned} \sum_{k=1}^{n} (x-x_k)^2 A_{0k}(x) &= x^2 - 2x \sum_{k=1}^{n} x_k A_{0k}(x) + \sum_{k=1}^{n} x_k^2 A_{0k}(x) \\ &= 2x \left[x - \sum_{k=1}^{n} x_k A_{0k}(x) \right] - \left[x^2 - \sum_{k=1}^{n} x_k^2 A_{0k}(x) \right] \\ &\leqslant 2R_{nm}(f_1, x) + R_{nm}(f_2, x) \leqslant 2r_{nm}(x), \end{aligned} \tag{4.12}$$

which, coupled with (2.38), yields (4.11).

Claim 2. We have the estimation

$$\sum_{|x - x_k| < d_k} d_k^2\, |A_{1k}(x)| \leqslant c_3 \frac{\|r_{nm}\| \ln^{10}[n(1+\|r_{nm}\|)]}{n}, \qquad x \in [-1, 1]. \tag{4.13}$$

In fact, by Lemma 4.1 it follows from (4.11) that

$$\|A_{1k}\| \leqslant 8 \frac{mn\, \|r_{nm}\|}{\sin \theta_k} \qquad (\sin \theta_k \neq 0), \quad k = 1, 2, ..., n, \tag{4.14}$$

and

$$\|A_{1k}\| \leqslant 4m^2 n^2\, \|r_{nm}\|, \qquad k = 1, 2, ..., n, \tag{4.15}$$

where $x_k = \cos\theta_k$, $k = 1, 2, ..., n$. Meanwhile by (4.11), (1.5), and (2.21)

$$2r_{nm}(x) \geqslant \sum_{k=1}^{n} (x - x_k)^2 B_{1k}(x)\, \ell_k(x)^m \geqslant c_1 \sum_{k=1}^{n} \frac{(x - x_k)^m\, \ell_k(x)^m}{d_k^{m-2}}.$$

This implies

$$|(x - x_k)\, \ell_k(x)| \leqslant c_4(\|r_{nm}\|\, d_k^{m-2})^{1/m} \leqslant c_5\, \|r_{nm}\|^{1/m}, \qquad k = 1, 2, ..., n.$$

Again applying Lemma 4.1 gives

$$|\ell_k(x)| \leqslant c_6 n^2\, \|r_{nm}\|^{1/m}, \qquad k = 1, 2, ..., n.$$

By a deep estimation of Erdős [2, Theorem 3] we obtain ($\theta_0 = 0$, $\theta_{n+1} = \pi$)

$$|\theta_{k+1} - \theta_k| \leqslant c_7 \frac{(\ln n) \ln(n\, \|r_{nm}\|)}{n}, \qquad k = 0, 1, ..., n,$$

and the maximum number K_n of the set $\{k: |x_k - x| < d_k\}$ when x runs over the interval $[-1, 1]$ satisfies

$$K_n \leqslant c_8 (\ln n) \ln(n\, \|r_{nm}\|). \tag{4.16}$$

Since for $1 \leqslant k \leqslant n-1$

$$\begin{aligned}
&|\cos\theta_{k+1} - \cos\theta_k| \\
&\quad = \left| 2 \sin\left(\theta_k + \frac{\theta_{k+1} - \theta_k}{2}\right) \sin\frac{\theta_{k+1} - \theta_k}{2} \right| \\
&\quad \leqslant \left| (\theta_{k+1} - \theta_k) \sin\left(\theta_k + \frac{\theta_{k+1} - \theta_k}{2}\right) \right| \\
&\quad = \left| (\theta_{k+1} - \theta_k) \left(\sin\theta_k \cos\frac{\theta_{k+1} - \theta_k}{2} + \cos\theta_k \sin\frac{\theta_{k+1} - \theta_k}{2} \right) \right| \\
&\quad \leqslant (\theta_{k+1} - \theta_k)(\sin\theta_k + \theta_{k+1} - \theta_k),
\end{aligned}$$

we have

$$d_k \leqslant c_9 \frac{(\ln n)(\ln n\, \|r_{nm}\|)}{n} \left(\sin\theta_k + \frac{(\ln n)(\ln n\, \|r_{nm}\|)}{n} \right),$$

$$k = 1, 2, ..., n. \tag{4.17}$$

If

$$\sin \theta_k \geqslant \frac{(\ln n)(\ln n \, \|r_{nm}\|)}{n},$$

then by (4.14) and (4.17)

$$d_k^2 \, |A_{1k}(x)| \leqslant c_{10} \left[\frac{\sin \theta_k (\ln n)(\ln n \, \|r_{nm}\|)}{n} \right]^2 \frac{n \, \|r_{nm}\|}{\sin \theta_k}$$

$$\leqslant c_3 \, \frac{\|r_{nm}\| \, \ln^4[n(1+\|r_{nm}\|)]}{n}.$$

If

$$\sin \theta_k < \frac{(\ln n)(\ln n \, \|r_{nm}\|)}{n},$$

then by (4.15) and (4.17)

$$d_k^2 \, |A_{1k}(x)| \leqslant c_{10} \left[\frac{(\ln n)(\ln n \, \|r_{nm}\|)}{n} \right]^4 n^2 \, \|r_{nm}\|$$

$$\leqslant c_3 \, \frac{\|r_{nm}\| \, \ln^8[n(1+\|r_{nm}\|)]}{n^2}.$$

This by (4.16) proves Claim 2.

Now for the proof of (4.9) we have

$$R_{nm}(P, x) = \left| \sum_{j=1}^{m-1} \sum_{k=1}^{n} P^{(j)}(x_k) \, A_{jk}(x) \right|$$

$$\leqslant \sum_{j=1}^{m-1} \left| \sum_{k=1}^{n} P^{(j)}(x_k) \, A_{jk}(x) \right| := \sum_{j=1}^{m-1} S_j.$$

We separate the three cases when $j=1$, $j=2$, and $j \geqslant 3$ (if $m=2$ then only the first case can occur).

By the mean value theorem for the derivative it follows from (2.36) and (4.11) that

$$S_1 \leqslant \left| P'(x) \sum_{k=1}^{n} A_{1k}(x) \right| + \left| \sum_{k=1}^{n} [P'(x) - P'(x_k)]\, A_{1k}(x) \right|$$

$$= |P'(x)|\, R_{nm}(f_1, x) + \left| \sum_{k=1}^{n} P''(\xi_k)(x - x_k)\, A_{1k}(x) \right|$$

$$\leqslant \|P'\|\, R_{nm}(f_1, x) + \|P''\| \sum_{k=1}^{n} (x - x_k)\, A_{1k}(x) \leqslant 3\, \|P\|^*\, r_{nm}(x).$$

Similarly, by (2.36), (4.11), and (4.12) we get

$$\left| \sum_{k=1}^{n} A_{2k}(x) \right| = \left| \frac{1}{2} \sum_{k=1}^{n} (x - x_k)^2\, A_{0k}(x) - \sum_{k=1}^{n} (x - x_k)\, A_{1k}(x) \right| \leqslant 3 r_{nm}(x)$$

and hence by (2.22) and (4.11)

$$S_2 \leqslant \|P''\| \left| \sum_{k=1}^{n} A_{2k}(x) \right| + \|P'''\| \sum_{k=1}^{n} |(x - x_k)\, A_{2k}(x)|$$

$$\leqslant \|P''\| \left| \sum_{k=1}^{n} A_{2k}(x) \right| + 2c_2\, \|P'''\| \sum_{k=1}^{n} (x - x_k)\, A_{1k}(x)$$

$$\leqslant (3 + 4c_2)\, \|P\|^*\, r_{nm}(x).$$

As for $j \geqslant 3$ by (1.5), (2.21), (2.22), and (4.13)

$$S_j \leqslant \|P^{(j)}\| \sum_{k=1}^{n} |A_{jk}(x)|$$

$$= \|P^{(j)}\| \left[\sum_{|x - x_k| \geqslant d_k} \left| \frac{1}{j!} (x - x_k)^j\, B_{jk}(x)\, \ell_k(x)^m \right| + \sum_{|x - x_k| < d_k} |A_{jk}(x)| \right]$$

$$\leqslant \|P^{(j)}\| \left[\frac{1}{c_1 j!} \sum_{|x - x_k| \geqslant d_k} |(x - x_k)^j\, B_{1k}(x)\, \ell_k(x)^m| \right.$$

$$\left. + c_2 \sum_{|x - x_k| < d_k} d_k^{j-1}\, |A_{1k}(x)| \right]$$

$$\leqslant c_{11}\, \|P^{(j)}\| \left[\sum_{|x - x_k| \geqslant d_k} (x - x_k)\, A_{1k}(x) + \sum_{|x - x_k| < d_k} d_k^2\, |A_{1k}(x)| \right]$$

$$\leqslant c\, \|P\|^* \left\{ r_{nm}(x) + \frac{\|r_{nm}\| \ln^{10}[n(1 + \|r_{nm}\|)]}{n} \right\}.$$

The second conclusion directly follows from (4.9). ∎

Then according to the Banach theorem we immediately obtain

THEOREM 4.2. *Let* $m_{kn} \equiv m$ *be an even integer. Then* (4.10) *holds for all* $f \in C[-1, 1]$ *if and only if*

$$\|H_{nm}\| = \left\| \sum_{k=1}^{n} |A_{0k}| \right\| = \bigcirc(1) \tag{4.18}$$

and (4.10) *holds for* $f = f_i$, $i = 1, 2$.

A simple and useful condition of convergence for Hermite–Fejeŕ-type interpolation is stated as follows.

THEOREM 4.3. *Let* $m_{k_m} \equiv m$ *be an even integer. If* (4.18) *is true and*

$$\lim_{n \to \infty} \left\| \sum_{k=1}^{n} |A_{1k}| \right\| = 0, \tag{4.19}$$

then (4.10) *holds for all* $f \in C[-1, 1]$.

Proof. Since (4.20) by (2.22) implies (4.19), Theorem 4.3 follows according to the following theorem. ▌

THEOREM D [14, Statement 2.1]. *Let* $m_{kn} \equiv m$ *be an even integer. If* (4.18) *is true and*

$$\lim_{n \to \infty} \left\| \sum_{j=1}^{m-1} \sum_{k=1}^{n} |A_{jk}| \right\| = 0, \tag{4.20}$$

then (4.10) *holds for all* $f \in C[-1, 1]$.

5. GRÜNWALD-TYPE THEOREMS

In [6] we proved a theorem of Grünwald type for Hermite–Fejér interpolation of higher order, which is a generalization of [3] given by Grünwald for $m = 2$.

THEOREM E [6, Theorem]. *Let* $m_{kn} \equiv m$ *be an even integer. If for fixed positive numbers* ρ_1 *and* n_0

$$\begin{aligned} &B_{0k}(x) \geqslant \rho_1 |B_{jk}(x)|, \\ &\qquad |x| \leqslant 1, \qquad j = 1, 2, ..., m-1, \quad k = 1, 2, ..., n, \quad n \geqslant n_0, \end{aligned} \tag{5.1}$$

then (4.10) *holds for all* $f \in C[-1, 1]$.

Later, Vértesi improved this result:

THEOREM F [14, Theorem 2.3]. *Let $m_{kn} \equiv m$ be an even integer. Let $I_{1n}(x)$ and $I_{2n}(x)$ be two disjoint subsets of the set $\{1, 2, ..., n\}$ with $I_{1n}(x) \cup I_{2n}(x) = \{1, 2, ..., n\}$. If for fixed positive numbers ρ_2 and n_0*

$$B_{0k}(x) \geqslant \rho_2 \, |B_{jk}(x)|, \qquad |x| \leqslant 1, \quad j = 1, 2, ..., m-1, \quad k \in I_{1n}(x), \quad n \geqslant n_0, \tag{5.2}$$

and

$$\lim_{n\to\infty} \Bigg\| \sum_{k \in I_{2n}(x)} |x - x_k|^\delta \, |A_{0k}(x)| \Bigg\| = 0, \qquad \forall \delta > 0, \tag{5.3}$$

$$\Bigg\| \sum_{k \in I_{2n}(x)} |A_{0k}(x)| \Bigg\| \leqslant C < \infty,$$

$$\lim_{n\to\infty} \Bigg\| \sum_{k \in I_{2n}(x)} |A_{jk}(x)| \Bigg\| = 0, \qquad j = 1, 2, ..., m-1, \tag{5.4}$$

$$\Bigg\| \sum_{k \in I_{2n}(x)} |B_{jk}(x) \, \ell_k(x)^m| \Bigg\| \leqslant C, \qquad j = 1, 2, ..., m-1, \tag{5.5}$$

then (4.10) *holds for all $f \in C[-1, 1]$.*

Although the conditions of this theorem seem to be complicated, practically this theorem is useful and convenient by using it the conditions of convergence of Hermite–Fejér-type interpolation based on the zeros of the Jacobi polynomials are derived in [14]. For the related papers we refer to a good survey paper [15] given recently by Vértesi.

Using Theorem 2.1 in this section we will essentially simplify and improve Theorems E and F. In view of Theorem 5.1 below it is natural to renew the definition of ρ-normality (in [6] we defined it by (5.1)).

DEFINTION. Let $m_{kn} \equiv m$ be an even integer. X is said to be ρ-normal if for fixed positive numbers ρ and n_0

$$B_{0k}(x) \geqslant \rho B_{1k}(x), \qquad |x| \leqslant 1, \quad k = 1, 2, ..., n, \quad n \geqslant n_0. \tag{5.6}$$

The main result of this section is as follows.

THEOREM 5.1. *Let $m_{kn} \equiv m$ be an even integer. Let $I_{1n}(x)$ and $I_{2n}(x)$ be two disjoint subsets of the set $\{1, 2, ..., n\}$ with $I_{1n}(x) \cup I_{2n}(x) = \{1, 2, ..., n\}$. If for fixed positive numbers ρ and n_0*

$$B_{0k}(x) \geqslant \rho B_{1k}(x), \qquad |x| \leqslant 1, \quad k \in I_{1n}(x), \quad n \geqslant n_0, \tag{5.7}$$

and

$$\lim_{n \to \infty} \left\| \sum_{k \in I_{2n}(x)} |x - x_k| \, |A_{0k}(x)| \right\| = 0, \tag{5.8}$$

$$\left\| \sum_{k \in I_{2n}(x)} |A_{0k}(x)| \right\| \leqslant C < \infty, \tag{5.9}$$

$$\lim_{n \to \infty} \left\| \sum_{k \in I_{2n}(x)} |A_{1k}(x)| \right\| = 0, \tag{5.10}$$

$$\left\| \sum_{k \in I_{2n}(x)} B_{1k}(x) \, \ell_k(x)^m \right\| \leqslant C, \tag{5.11}$$

then (4.10) *holds for all $f \in C[-1, 1]$.*

As a special case of Theorem 5.2 ($I_{2n}(x) \equiv \varnothing$) we state the second main result.

COROLLARY 5.1. *Let $m_{kn} \equiv m$ be an even integer. If X is ρ-normal then* (4.10) *holds for all $f \in C[-1, 1]$.*

Let

$$\Delta_n(x) := \frac{(1 - x^2)^{1/2}}{n} + \frac{1}{n^2}$$

and $x_0 = 1$, $x_{n+1} = -1$, $x_k = \cos \theta_k$, $k = 0, 1, ..., n+1$.

LEMMA 5.1. *If*

$$\theta_{k+1} - \theta_k \geqslant \frac{c_2}{n}, \qquad k = 1, 2, ..., n-1, \tag{5.12}$$

and

$$\theta_{k+1} - \theta_k \leqslant \frac{c_3}{n}, \qquad k = 0, 1, ..., n, \tag{5.13}$$

then

$$|x_\nu - x_k| \sim \frac{|\nu-k| \min\{\nu+k, 2n+2-\nu-k\}}{n^2},$$

$$\nu \neq k, \quad 1 \leqslant \nu, k \leqslant n, \tag{5.14}$$

$$|x_k - x_{k+1}| \sim \frac{\min\{k, n-k\}}{n^2} \sim \Delta_n(x_k), \qquad k = 1, 2, ..., n-1, \tag{5.15}$$

and

$$d_k \sim \frac{\min\{k, n+1-k\}}{n^2} \sim \Delta_n(x_k), \qquad k = 1, 2, ..., n. \tag{5.16}$$

Proof. First, clearly (5.12) and (5.13) imply

$$|\theta_\nu - \theta_k| \sim \frac{|\nu-k|}{n}. \tag{5.17}$$

On the other hand, by (5.13)

$$\theta_\nu + \theta_k = \theta_\nu - \theta_0 + \theta_k - \theta_0 \leqslant \frac{c_3(\nu+k)}{n}$$

and by (5.12)

$$\theta_\nu + \theta_k \geqslant \theta_\nu - \theta_1 + \theta_k - \theta_1 \geqslant \frac{c_2(\nu+k-2)}{n} \geqslant \frac{c_2(\nu+k)}{3n},$$

since $\nu \neq k$ and hence $\nu + k \geqslant 3$. That is

$$\theta_\nu + \theta_k \sim \frac{\nu+k}{n}. \tag{5.18}$$

Similarly we have

$$2\pi - \theta_\nu - \theta_k = 2\theta_{n+1} - \theta_\nu - \theta_k \sim \frac{2n+2-\nu-k}{n}. \tag{5.19}$$

Thus (5.14) may be obtained from (5.17)–(5.19) and the following three relations:

$$|x_\nu - x_k| = |\cos\theta_\nu - \cos\theta_k| = \left|2\sin\frac{\theta_\nu+\theta_k}{2}\sin\frac{\theta_\nu-\theta_k}{2}\right|,$$

$$\sin\frac{\theta_\nu-\theta_k}{2} \sim |\theta_\nu - \theta_k|,$$

$$\sin\frac{\theta_\nu+\theta_k}{2} \sim \begin{cases}\theta_\nu+\theta_k, & \nu+k\leqslant n,\\ 2\pi-\theta_\nu-\theta_k, & \nu+k>n.\end{cases}$$

Next, the first relation in (5.15) follows from (5.14). The second one follows from the relations:

$$\Delta_n(x_k) = \frac{\sin\theta_k}{n} + \frac{1}{n^2} \sim \frac{\min\{\theta_k, \pi-\theta_k\}}{n} + \frac{1}{n^2} \sim \frac{\min\{k, n+1-k\}}{n^2}.$$

Finally, (5.16) is a direct consequence of (5.15). ▮

LEMMA 5.2. *Let* $m_{kn} \equiv m$ *be even. If* (5.12), (5.13), *and*

$$|\ell'_k(x_k)| = \bigcirc(1)\,\Delta_n(x_k)^{-1}, \qquad k=1, 2, ..., n, \tag{5.20}$$

hold, then

$$|b_{ik}| = \bigcirc(1)\,\Delta_n(x_k)^{-i}, \qquad k=1, 2, ..., n, \quad i=0, 1, ..., \tag{5.21}$$

and

$$b_{ik} \sim \Delta_n(x_n)^{-i}, \qquad k=1, 2, ..., n, \quad i=0, 2, 4, ... \tag{5.22}$$

Furthermore, if $m-p$ *is odd then*

$$B_{pk}(x) \geqslant c_4\,|B_{ik}(x)|, \qquad x\in\mathbb{R}, \quad p<i\leqslant m-1, \quad 1\leqslant k\leqslant n. \tag{5.23}$$

Moreover, if we replace $\Delta_n(x_k)$ *by* $(1-x_k^2)^{1/2}/n$ *and* (5.12) *by*

$$\theta_{k+1}-\theta_k \geqslant \frac{c_2}{n}, \qquad k=0, 1, ..., n, \tag{5.24}$$

then the above conclusions remain true.

Proof. First let us prove the first part of the lemma. Assume without loss of generality that $k \leqslant n/2$. Then for $i \geqslant 2$ by (5.14) and (5.15)

$$
\begin{aligned}
|a_{ik}| &= \bigcirc(1) \sum_{v \neq k} \left(\frac{n^2}{|v-k| \min\{v+k, 2n+2-v-k\}} \right)^i \\
&= \bigcirc(1) \left[\sum_{v \leqslant 3n/4,\, v \neq k} \left(\frac{n^2}{k\,|v-k|} \right)^i + \sum_{v > 3n/4} \left(\frac{n^2}{k(2n+2-v-k)} \right)^i \right] \\
&= \bigcirc(1) \left(\frac{n^2}{k} \right)^i = \bigcirc(1)\, \Delta_n(x_k)^{-i}.
\end{aligned}
$$

Since $a_{1k} = -\ell_k'(x_k)$, we have by (5.20)

$$|a_{ik}| = \bigcirc(1)\, \Delta_n(x_k)^{-i}, \qquad k = 1, 2, ..., n, \quad i = 1, 2, ...,$$

which coupled with (2.20) gives (5.21).

On the other hand, by (1.4), (2.21), and (5.16) for even i

$$b_{ik} \geqslant c d_k^{-i} \sim \Delta_n(x_k)^{-i},$$

which coupled with (5.21) yields (5.22).

Let us prove (5.23). According to (5.21) and (5.16) we can conclude that, assuming without loss of generality $c_5 \geqslant 1$,

$$|b_{vk}| \leqslant c_5 d_k^{-v}, \qquad v = 0, 1, ..., m-1, \quad k = 1, 2, ..., n. \tag{5.25}$$

Choose $h = 1/(2mc_5)$. We distinguish two cases.

Case 1. $|x - x_k| \geqslant h d_k$. In this case by (2.21)

$$B_{pk}(x) \geqslant c h^{i-p}\, |B_{ik}(x)| \geqslant c h^{m-1}\, |b_{ik}(x)|.$$

Case 2. $|x - x_k| < h d_k$. In this case by (1.4) and (5.25)

$$B_{pk}(x) \geqslant 1 - \left| \sum_{v=1}^{m-p-1} b_{vk}(x - x_k)^v \right| \geqslant 1 - c_5 \sum_{v=1}^{m-p-1} h^v \geqslant 1 - mhc_5 = \tfrac{1}{2}$$

and

$$B_{ik}(x) \leqslant 1 + \left| \sum_{v=1}^{m-i-1} b_{vk}(x - x_k)^v \right| \leqslant 1 + c_5 \sum_{v=1}^{m-i-1} h^v \leqslant 1 + mhc_5 = \tfrac{3}{2}.$$

Thus in this case $B_{pk}(x) \geqslant \frac{1}{3} |B_{ik}(x)|$.

Then (5.23) follows if we put $c_4 = \min\{ch^{m-1}, \frac{1}{3}\}$.

Next, to prove the second part of the lemma we note that in this case

$$\Delta_n(x_k) \sim \frac{(1-x_k^2)^{1/2}}{n}, \qquad k=1, 2, ..., n.$$

Hence the conclusions follow. ▮

Remark 5.1. In general the condition (5.20) in Lemma 5.2 cannot be dropped. For example put $n=2r-1 (r \geqslant 2)$ and

$$\theta_r = \frac{\pi}{3}, \qquad \theta_{r-k} = \left(\frac{1}{3} - \frac{k}{3r}\right)\pi,$$

$$\theta_{r+k} = \left(\frac{1}{3} + \frac{2k}{3r}\right)\pi, \qquad k=1, 2, ..., r-1.$$

Although (5.13) and (5.24) hold, by an elementary calculation we can get

$$b_{1r} = ma_{1r} = -m\ell'_r(x_r) \geqslant c_6 \Delta_n(x_r)^{-1} \ln n.$$

LEMMA 5.3. *If for* $\xi = \cos \tau$, $\tau \in [0, \tau]$,

$$\min_{0 \leqslant k \leqslant n+1} |\tau - \theta_k| \geqslant d \max_{0 \leqslant k \leqslant n} (\theta_{k+1} - \theta_k), \qquad d > 0, \tag{5.26}$$

then

$$|\xi - x_k| \geqslant \frac{d^2}{16} d_k, \qquad k=1, 2, ..., n. \tag{5.27}$$

Proof. It is easy to check that

$$\sin \theta \leqslant \theta, \qquad 0 \leqslant \theta \leqslant \pi; \qquad \sin \theta \geqslant \frac{\theta}{4}, \qquad 0 \leqslant \theta \leqslant \frac{3\pi}{4}. \tag{5.28}$$

Now suppose without loss of generality that $\tau \leqslant \pi/2$ (The case $\tau > \pi/2$ leads to entirely analogous, symmetric discussion). Let k, $1 \leqslant k \leqslant n$, be fixed. Then by (5.28)

$$|\xi - x_k| = |\cos \tau - \cos \theta_k| = \left|2 \sin \frac{\tau + \theta_k}{2} \sin \frac{\tau - \theta_k}{2}\right| \geqslant \frac{1}{32} |\tau^2 - \theta_k^2| \tag{5.29}$$

and

$$|x_{k+1}-x_k|=|\cos\theta_{k+1}-\cos\theta_k|$$
$$=\left|2\sin\frac{\theta_{k+1}+\theta_k}{2}\sin\frac{\theta_{k+1}-\theta_k}{2}\right|\leqslant\frac{1}{2}(\theta_{k+1}^2-\theta_k^2). \tag{5.30}$$

Clearly by (5.26)

$$|\tau-\theta_k|\geqslant d\,|\theta_k-\theta_{k-1}|, \qquad |\tau-\theta_k|\geqslant d\,|\theta_k-\theta_{k+1}|.$$

Again by (5.26) we get $d\leqslant 1/2$ and hence

$$\theta_{k+1}-\theta_k\leqslant\frac{\tau-\theta_0}{d}\leqslant\frac{\tau+(1-2d)\,\theta_k}{d}=\frac{\tau+\theta_k}{d}-2\theta_k,$$

which implies

$$\tau+\theta_k\geqslant d(\theta_k+\theta_{k+1})\geqslant d(\theta_k+\theta_{k-1}).$$

To sum up by (5.29) and (5.30)

$$|\xi-x_k|\geqslant\frac{1}{32}|\tau^2-\theta_k^2|\geqslant\frac{d^2}{32}\max\{|\theta_k^2-\theta_{k-1}^2|,|\theta_k^2-\theta_{k+1}^2|\}$$
$$\geqslant\frac{d^2}{16}\max\{|x_k-x_{k-1}|,|x_k-x_{k+1}|\}\geqslant\frac{d^2}{16}d_k. \quad \blacksquare$$

Following the line of the proof of Theorem 1 in [1] given by Erdős and Turán we can prove

LEMMA 5.4. *Let $m_{kn}\equiv m$ be even. If $m-j$ is odd and*

$$|B_{jk}(x)\,\ell_k(x)^m|\leqslant c_8, \quad x\in[-1,1], \quad k=1,2,...,n, \quad n=1,2,..., \tag{5.31}$$

then (5.12), (5.13), *and* (5.20)–(5.23) *hold.*

Proof. By Lemma 5.2 it suffices to show (5.12), (5.13), and (5.20). First by Rolle's theorem and the Bernstein inequality for $1\leqslant k\leqslant n-1$

$$\frac{1}{|\theta_{k+1}-\theta_k|}=\left|\frac{B_{jk}(\cos\theta_k)\,\ell_k(\cos\theta_k)^m-B_{jk}(\cos\theta_{k+1})\,\ell_k(\cos\theta_{k+1})^m}{\theta_k-\theta_{k+1}}\right|$$
$$=\left|\frac{d[B_{jk}(\cos\theta)\,\ell_k(\cos\theta)^m]}{d\theta}\right|_{\theta=\theta'}\leqslant c_9(mn-1),$$

which is equivalent to (5.12).

Next let us prove (5.13). Using the same argument as that of Theorem 1 in [1] let

$$\max_{0 \leqslant k \leqslant n} (\theta_{k+1} - \theta_k) = \theta_{r+1} - \theta_r = \frac{2c_2 D_n^*}{n}$$

and

$$\tau = \frac{\theta_{r+1} + \theta_r}{2}, \qquad \xi = \cos \tau, \tag{5.32}$$

where c_2 is given in (5.12). To prove (5.13) it is enough to show

$$D_n^* \leqslant c_{10}. \tag{5.33}$$

To this end put

$$\phi(\theta) = \frac{1}{n^2} \left(\frac{\sin n \dfrac{\theta+\tau}{2}}{\sin \dfrac{\theta+\tau}{2}} \right)^2 + \frac{1}{n^2} \left(\frac{\sin n \dfrac{\theta-\tau}{2}}{\sin \dfrac{\theta-\tau}{2}} \right)^2. \tag{5.34}$$

In [1, (23), (25), and (26)] it is proved that

$$\phi(\tau) \geqslant 1, \tag{5.35}$$

$$|\phi(\theta)| \leqslant \frac{9\pi^2}{2n^2} \left[\frac{1}{(\theta+\tau)^2} + \frac{1}{(\theta-\tau)^2} \right], \tag{5.36}$$

and

$$\phi(\theta) \equiv \sum_{k=1}^{n} \phi(\theta_k)\, \ell_k(\cos \theta). \tag{5.37}$$

By Lemma 5.3 it follows from (5.32) that $(d = 1/2)$

$$|\xi - x_k| \geqslant \frac{d_k}{64}, \qquad k = 1, 2, ..., n. \tag{5.38}$$

Then noting that $B_{m-1,k}(x) \equiv 1$ and using (5.31), (2.21), and (5.38) we get

$$c_8 \geqslant |B_{jk}(\xi)\, \ell_k(\xi)^m| \geqslant c \left| \frac{\xi - x_k}{d_k} \right|^{m-j-1} |\ell_k(\xi)^m| \geqslant \frac{c}{(64)^{m-j-1}} |\ell_k(\xi)^m|$$

and hence

$$|\ell_k(\xi)| \leqslant c_{11}, \qquad k=1, 2, ..., n, \quad n=1, 2, ... \tag{5.39}$$

By virtue of (5.35)–(5.37) and (5.39) (see [1, (27)])

$$\begin{aligned} 1 \leqslant \phi(\tau) &= \sum_{k=1}^{n} \phi(\theta_k)\, \ell_k(\cos\tau) \\ &\leqslant \frac{9\pi^2}{2n^2} c_{11} \sum_{k=1}^{n} \left[\frac{1}{(\theta_k+\tau)^2} + \frac{1}{(\theta_k-\tau)^2} \right] := \frac{9c_{11}\pi^2}{2n^2} S. \end{aligned} \tag{5.40}$$

In [1, (29a) and (29b)] it is shown that

$$S \leqslant \frac{3}{c_2^2} \cdot \frac{n^2}{D_n^* - 1},$$

which together with (5.40) gives (5.33).

Finally to prove (5.20) by (2.21) and (5.31) we obtain

$$|(x-x_k)^{m-j-1}\, \ell_k(x)^m| \leqslant \frac{c_8}{c}\, d_k^{m-j-1}.$$

By the Markov inequality [4, (3.4), p. xxix] and (5.16)

$$\begin{aligned} |[(x-x_k)^{m-j-1}\, \ell_k(x)^m]^{(m-j)}_{x=x_k}| &= \bigcirc(1)\, d_k^{m-j-1} \Delta_n(x_k)^{j-m} \\ &= \bigcirc(1)\, \Delta_n(x_k)^{-1}. \end{aligned}$$

Applying the Newton–Leibniz rule the above relation becomes

$$|\ell'_k(x_k)| = \bigcirc(1)\, \Delta_n(x_k)^{-1},$$

which proves (5.20). ∎

Remark 5.2. I believe that Lemma 5.4 remains true even if $m-j$ is even. That is the following

CONJECTURE 5.1. *Let* $m_{kn} \equiv m$ *be even. If* $m-j$ *is even and* (5.31) *is true then* (5.12), (5.13), *and* (5.20)–(5.23) *hold.*

Up to now we know that this conjecture is true for the case when $m=2$ and $j=0$. In fact, in this case (5.12) and (5.13) can be found in [1, Sec. 4]. To prove (5.20) we note that (5.31) implies by the Markov inequality

$$|[B_{0k}(x)\, \ell_k(x)^2]''_{x=x_k}| = \bigcirc(1)\, \Delta_n(x_k)^{-2}.$$

Since $B_{0k}(x) = 1 - 2\ell'_k(x_k)(x - x_k)$, using (2.54) we obtain

$$|[B_{0k}(x)\,\ell_k(x)^2]''_{x=x_k}| = 2[3\ell'_k(x_k)^2 - \ell''_k(x_k)] \geqslant 4\ell'_k(x_k)^2.$$

Hence (5.20) follows.

As an immediate consequence of Lemma 5.4 we state

COROLLARY 5.2. *Let $m_{kn} \equiv m$ be even. If $m - j$ is odd and*

$$\sum_{k=1}^{n} |B_{jk}(x)\,\ell_k(x)^m| = \bigcirc(1), \tag{5.41}$$

then

$$\sum_{k=1}^{n} |B_{ik}(x)\,\ell_k(x)^m| = \bigcirc(1), \qquad j < i \leqslant m-1. \tag{5.42}$$

Proof of Theorem 5.1. Using the identity $\sum_{k=1}^{n} A_{0k}(x) \equiv 1$ it follows from (5.7) and (5.9) that

$$\begin{aligned}\sum_{k \in I_{1n}(x)} B_{1k}(x)\,\ell_k(x)^m &\leqslant \frac{1}{\rho} \sum_{k \in I_{1n}(x)} B_{0k}(x)\,\ell_k(x)^m = \frac{1}{\rho} \sum_{k \in I_{1n}(x)} A_{0k}(x)\\ &= \frac{1}{\rho}\left[1 - \sum_{k \in I_{2n}(x)} A_{0k}(x)\right] \leqslant \frac{1+M}{\rho},\end{aligned}$$

which coupled with (5.11) yields

$$\sum_{k=1}^{n} B_{1k}(x)\,\ell_k(x)^m \leqslant \frac{1+M}{\rho} + M.$$

Applying Lemma 5.4 it follows from (5.23) that

$$B_{1k}(x) \geqslant c_4\,|B_{ik}(x)|, \qquad x \in \mathbb{R}, \quad 2 \leqslant i \leqslant m-1, \quad 1 \leqslant k \leqslant n.$$

Using this inequality (5.7) and (5.11) imply (5.2) and (5.5), respectively. Meanwhile, (5.10) by (2.22) implies (5.4). On the other hand, using Hölder's inequality (5.3) follows from (5.8) and (5.9). Thus all the assumptions of Theorem F are satisfied and we can apply Theorem F to conclude that (4.10) holds for all $f \in C[-1, 1]$. ∎

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